邊界元素法應用於潮流引起擴散

在水平面內取直角座標系(x, y), z 軸垂直向上，潮流在 x, y 方向分量為 u, v。不可壓縮流的連續方程式如下

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (1)$$

利用 Navier-Stokes 方程式描述流的運動

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial P}{\partial x}+fv-\frac{1}{\rho}\frac{\partial F_x}{\partial z} \qquad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y-\frac{1}{\rho}\frac{\partial P}{\partial y}-fu-\frac{1}{\rho}\frac{\partial F_y}{\partial z} \qquad (3)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z-\frac{1}{\rho}\frac{\partial P}{\partial z}-\frac{1}{\rho}\frac{\partial F_x}{\partial z} \qquad (4)$$

ρ 為海水密度，P 為壓力，f 柯氏力，X、Y、Z 為外力，F_x、F_y、F_z 為摩擦力。

假定為淺水條件，且 X=Y=0，Z=-g，(4)式左邊的水粒子於 z 方向加速度與重力加速度相比為小，可忽略不計，即

$$0=-g-\frac{1}{\rho}\frac{\partial P}{\partial z}-\frac{1}{\rho}\frac{\partial F_x}{\partial z}\cdots... \quad (5)$$

忽略垂直方向摩擦力變化時，上式可改寫為

$$0=-g-\frac{1}{\rho}\frac{\partial P}{\partial z}$$

即

$$dP=-\rho g dz$$

由海底（z = -h）積分到海面（z = ζ，水位的上昇量）時，得

$$P=\int_{-h}^{\zeta}\rho g dz=P_0-g\int_{-h}^{\zeta}\rho dz\cdots... \quad (6)$$

P_0為大氣壓力，x、 y 軸方向的壓力分量為

$$\frac{\partial P}{\partial x}=\frac{\partial P_0}{\partial x}-g\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho dz$$

$$\frac{\partial P}{\partial y}=\frac{\partial P_0}{\partial y}-g\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho dz$$

假設壓力梯度變化小，得

$$\frac{\partial P_0}{\partial x}=\frac{\partial P_0}{\partial y}=0$$

即

$$\frac{\partial P}{\partial x}=-g\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho dz$$

$$\frac{\partial P}{\partial y}=-g\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho dz$$

因

$$g\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho dz=g\int_{-h}^{\zeta}\frac{\partial \rho}{\partial x}dz+g\rho|_{\zeta}\frac{\partial \zeta}{\partial x}-g\rho|_{\zeta}\frac{\partial(-h)}{\partial x}$$

假定海底地形變化緩和，可忽略$\partial(-h)/\partial x$及$\partial(-h)/\partial y$。

假定海水密度為定值，不隨水平方向變化，得

$$\frac{\partial \rho}{\partial x}=\frac{\partial \rho}{\partial y}=0$$

故得

$$\frac{\partial P}{\partial x}=\rho g\frac{\partial \zeta}{\partial x} \qquad (7)$$

$$\frac{\partial P}{\partial y}=\rho g\frac{\partial \zeta}{\partial y} \qquad (8)$$

(2)、(3)式可以改寫為

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-g\frac{\partial \zeta}{\partial x} \qquad (9)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-g\frac{\partial \zeta}{\partial y} \qquad (10)$$

為了簡化計算，將(9)、(10)式的非線性項忽略，得

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial x}=0 \qquad (11)$$

$$\frac{\partial v}{\partial t}+g\frac{\partial \zeta}{\partial y}=0 \qquad (12)$$

(11)、(12)為不考慮摩擦效應時的長波方程式。

將連續方程式對水深方向積分得

$$\int_{-h}^{\zeta}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]dz=-\int_{-h}^{\zeta}\frac{\partial w}{\partial z}dz=-w|_{\zeta}+w|_{-h}$$

即

$$\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]\int_{-h}^{\zeta}dz=-w|_{\zeta}+w|_{-h} \qquad (13)$$

$$w|_{\zeta}=\frac{d\zeta}{dt}=\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y} \qquad (14)$$

$$w|_{-h}=-\frac{dh}{dt}=-\frac{\partial h}{\partial t}-u\frac{\partial h}{\partial x}-v\frac{\partial h}{\partial y} \qquad (15)$$

將(14)、(15)式代入(13)式得

$$\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right](\zeta+h)=-\frac{\partial \zeta}{\partial t}-\frac{\partial h}{\partial t}-u\left(\frac{\partial \zeta}{\partial x}+\frac{\partial h}{\partial x}\right)-v\left(\frac{\partial \zeta}{\partial y}+\frac{\partial h}{\partial y}\right)$$

$$\frac{\partial \zeta}{\partial t}+\frac{\partial h}{\partial t}+\left\{\frac{\partial u}{\partial x}(\zeta+h)+u\left(\frac{\partial \zeta}{\partial x}+\frac{\partial h}{\partial x}\right)\right\}+\left\{\frac{\partial v}{\partial y}(\zeta+h)+v\left(\frac{\partial \zeta}{\partial y}+\frac{\partial h}{\partial y}\right)\right\}=0$$

由於海底為固定底床，

$$\frac{\partial h}{\partial t}=0$$

得

$$\frac{\partial\zeta}{\partial t}+\frac{\partial u\left(\zeta+h\right)}{\partial x}+\frac{\partial v\left(\zeta+h\right)}{\partial y}=0 \qquad (16)$$

將(11)式對 x 作偏微分，(12)式對 y 作偏微分得

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+g\frac{\partial\zeta}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)+\frac{\partial}{\partial x}\left(g\frac{\partial\zeta}{\partial x}\right)=\frac{\partial^2 u}{\partial x\partial t}+g\frac{\partial^2\zeta}{\partial x^2}=0 \qquad (17)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}+g\frac{\partial\zeta}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right)+\frac{\partial}{\partial y}\left(g\frac{\partial\zeta}{\partial y}\right)=\frac{\partial^2 v}{\partial y\partial t}+g\frac{\partial^2\zeta}{\partial y^2}=0 \qquad (18)$$

將(16)式對時間作偏微分，代入(11)、(12)、(17)、(18)式的條件得

$$\frac{\partial}{\partial x}\left(h\frac{\partial\zeta}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial\zeta}{\partial y}\right)-\frac{1}{g}\frac{\partial^2\zeta}{\partial t^2}=0 \qquad (19)$$

開放海域內無結構物存在時，潮汐引起的水位作$\zeta_o\exp(-i\sigma t)$形式的簡諧運動，σ 為角週頻率(=2π/T，T=週期)，ζ_o為水位振幅，水位為

$$\zeta_i(x,y;t)=-i\zeta_0\exp\left\{-i\left[k(x\cos\varpi+y\sin\varpi)+\sigma t\right]\right\}$$

海域受海底地形及結構物等影響引起水位變化ζ(x, y)可以下式表示

$$\zeta(x,y)=w(x,y)+\zeta_i(x,y) \qquad (20)$$

潮流方向為與 x 軸呈ϖ角度，週波數 k 可由下式求得

$$k^2=\sigma^2/gh$$

(19)式可改寫成

$$\frac{\partial}{\partial x}\left(h\frac{\partial\zeta}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial\zeta}{\partial y}\right)+k^2h\zeta_i=0$$

令 H=1-h，將(20)式代入上式可得

$$\nabla^2 w + k^2 w = H\nabla^2 w + \nabla H \cdot \nabla w - \alpha(x,y) \qquad (21)$$

$$\alpha = \left(\frac{\partial}{\partial x}\left(h\frac{\partial \zeta_i}{\partial x} \right) + \frac{\partial}{\partial y}\left(h\frac{\partial \zeta_i}{\partial y} \right) + k^2 h\zeta_i - k^2 w \right)$$

將(21)式乘以加權函數 G 並積分得

$$\int G(\nabla^2 w + k^2 w)d\Omega = \int GH\nabla^2 w d\Omega + \int G\nabla H \cdot \nabla w d\Omega - \int G\alpha(x,y)d\Omega$$

利用 Green 定理，上式可改寫成

$$\int w(\nabla^2 G + k^2 G)d\Omega = \int w\nabla \cdot (H\nabla G)d\Omega$$
$$+\int GH\frac{\partial w}{\partial n}d\Gamma - \int Hw\frac{\partial G}{\partial n}d\Gamma + \int w\frac{\partial G}{\partial n}d\Gamma - \int G\frac{\partial w}{\partial n}d\Gamma - \int G\alpha(x,y)d\Omega$$

上式左邊，加權函數 G 能滿足下式

$$\nabla^2 G + k^2 G = -\delta(Q-P)$$

其基本解為

$$G = \frac{i}{4}H_0^{(1)}(kr)$$

式中，$H_0^{(1)}$ 為第 1 類 0 次 Hankel 函數，故線性長波方程式的積分表示如下

$$\gamma w = \int w\nabla \cdot (H\nabla G)d\Omega$$
$$+\int GH\frac{\partial w}{\partial n}d\Gamma - \int Hw\frac{\partial G}{\partial n}d\Gamma + \int w\frac{\partial G}{\partial n}d\Gamma - \int G\frac{\partial w}{\partial n}d\Gamma - \int G\alpha(x,y)d\Omega$$

在邊界上 γ=0.5，在領域內 γ=1。

回應用邊界元素法解析海洋擴散